1 Motion in a straight line

The whole burden of philosophy seems to consist in this – from the phenomena of motions to investigate the forces of nature.

Isaac Newton

The language of motion

Throw a small object such as a marble straight up in the air and think about the words you could use to describe its motion from the instant just after it leaves your hand to the instant just before it hits the floor. Some of your words might involve the idea of direction. Other words might be to do with the position of the marble, its speed or whether it is slowing down or speeding up. Underlying many of these is time.

Direction

The marble moves as it does because of the gravitational pull of the earth. We understand directional words such as up and down because we experience this pull towards the centre of the earth all the time. The *vertical* direction is along the line towards or away from the centre of the earth.

In mathematics a quantity which has only size, or magnitude, is called a *scalar*. One which has both magnitude and a direction in space is called a *vector*.

Distance, position and displacement

The total *distance* travelled by the marble at any time does not depend on its direction. It is a scalar quantity.

Position and displacement are two vectors related to distance: they have direction as well as magnitude. Here their direction is up or down and you decide which of these is positive. When up is taken to be positive, down is negative.

The *position* of the marble is then its distance above a fixed origin, for example the distance above the place it first left your hand.

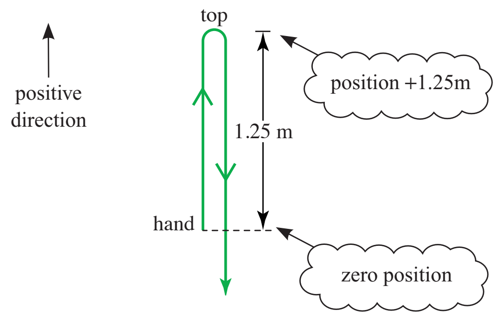


Figure 1.1

When it reaches the top, the marble might have travelled a distance of 1.25 m. Relative to your hand its position is then 1.25 m upwards or +1.25 m.

At the instant it returns to the same level as your hand it will have travelled a total distance of 2.5 m. Its *position*, however, is zero upwards.

A position is always referred to a fixed origin but a *displacement* can be measured from any position. When the marble returns to the level of your hand, its displacement is zero relative to your hand but −1.25 m relative to the top.

? What are the positions of the particles A, B and C in the diagram below?

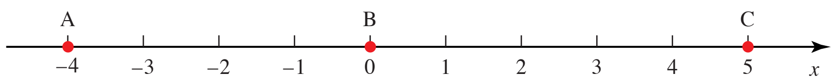


Figure 1.2

What is the displacement of B

(i) relative to A **(ii)** relative to C?

Diagrams and graphs

In mathematics, it is important to use words precisely, even though they might be used more loosely in everyday life. In addition, a picture in the form of a diagram or graph can often be used to show the information more clearly.

Figure 1.3 is a *diagram* showing the direction of motion of the marble and relevant distances. The direction of motion is indicated by an arrow. Figure 1.4 is a *graph* showing the position above the level of your hand against the time. Notice that it is *not* the path of the marble.

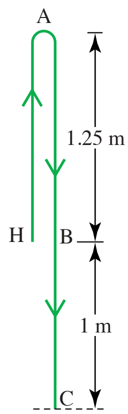


Figure 1.3

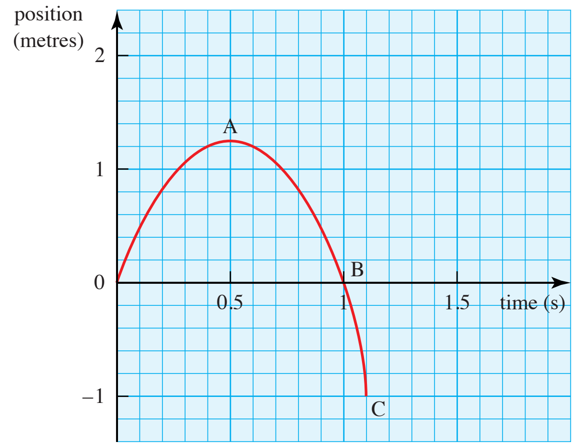


Figure 1.4

The graph in figure 1.4 shows that the position is negative after one second (point B). What does this negative position mean?

Note

When drawing a graph it is very important to specify your axes carefully. Graphs showing motion usually have time along the horizontal axis. Then you have to decide where the origin is and which direction is positive on the vertical axis. In this graph the origin is at hand level and upwards is positive. The time is measured from the instant the marble leaves your hand.

Notation and units

As with most mathematics, you will see in this book that certain letters are commonly used to denote certain quantities. This makes things easier to follow. Here the letters used are:

- s, h, x, y and z for position
- t for time measured from a starting instant
- u and v for velocity
- a for acceleration.

The S.I. (Système International d'Unités) unit for *distance* is the metre (m), that for *time* is the second (s) and that for *mass* the kilogram (kg). Other units follow from these so speed is measured in metres per second, written $\mathrm{m\,s^{-1}}$.

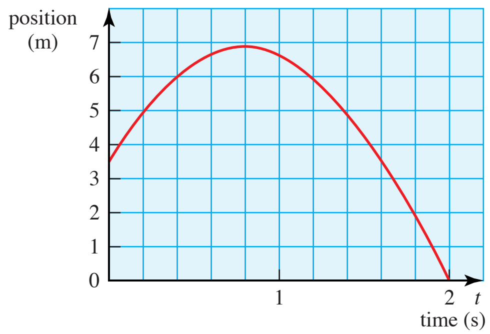
EXERCISE 1A

1 When the origin for the motion of the marble (see figure 1.3) is on the ground, what is its position

(i) when it leaves your hand?

(ii) at the top?

2 A boy throws a ball vertically upwards so that its position y m at time t is as shown in the graph.



(i) Write down the position of the ball at times $t = 0$, 0.4, 0.8, 1.2, 1.6 and 2.

(ii) Calculate the displacement of the ball relative to its starting position at these times.

(iii) What is the total distance travelled

(a) during the first 0.8 s **(b)** during the 2 s of the motion?

3 The position of a particle moving along a straight horizontal groove is given by $x = 2 + t(t - 3)$ for $0 \leqslant t \leqslant 5$ where x is measured in metres and t in seconds.

(i) What is the position of the particle at times $t = 0$, 1, 1.5, 2, 3, 4 and 5?

(ii) Draw a diagram to show the path of the particle, marking its position at these times.

(iii) Find the displacement of the particle relative to its initial position at $t = 5$.

(iv) Calculate the total distance travelled during the motion.

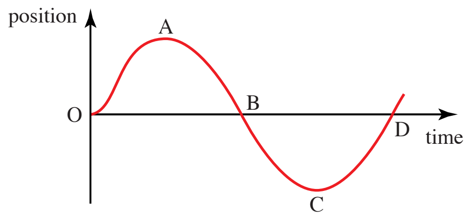
4 For each of the following situations sketch a graph of position against time. Show clearly the origin and the positive direction.

(i) A stone is dropped from a bridge which is 40 m above a river.

(ii) A parachutist jumps from a helicopter which is hovering at 2000 m. She opens her parachute after 10 s of free fall.

(iii) A bungee jumper on the end of an elastic string jumps from a high bridge.

5 The diagram is a sketch of the position–time graph for a fairground ride.



(i) Describe the motion, stating in particular what happens at O, A, B, C and D.

(ii) What type of ride is this?

Speed and velocity

Speed is a scalar quantity and does not involve direction. *Velocity* is the vector related to speed; its magnitude is the speed but it also has a direction. When an object is moving in the negative direction, its velocity is negative.

Amy has to post a letter on her way to college. The post box is 500 m east of her house and the college is 2.5 km to the west. Amy cycles at a steady speed of $10\,\mathrm{m\,s^{-1}}$ and takes 10 s at the post box to find the letter and post it.

Figure 1.5 shows Amy's journey using east as the positive direction. The distance of 2.5 km has been changed to metres so that the units are consistent.

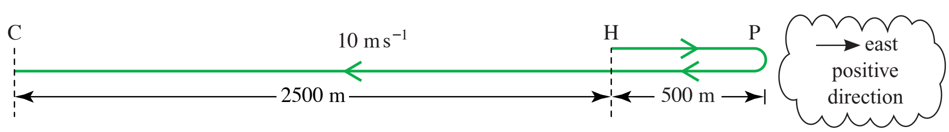


Figure 1.5

After she leaves the post box Amy is travelling west so her velocity is negative. It is $-10\,\mathrm{m\,s^{-1}}$.

The distances and times for the three parts of Amy's journey are:

	Distance	Time
Home to post box	500 m	$\frac{500}{10} = 50\,\text{s}$
At post box	0 m	10 s
Post box to college	3000 m	$\frac{3000}{10} = 300\,\text{s}$

These can be used to draw the position–time graph using home as the origin, as in figure 1.6.

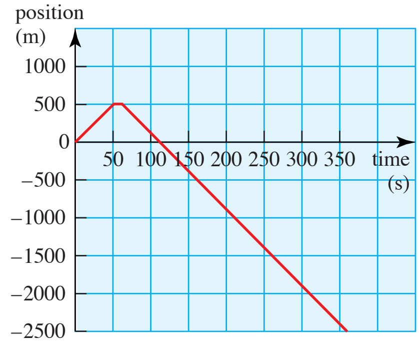


Figure 1.6

? Calculate the gradient of the three portions of this graph. What conclusions can you draw?

The velocity is the rate at which the position changes.

- Velocity is represented by the gradient of the position–time graph.

Figure 1.7 is the velocity–time graph.

Note

By drawing the graphs below each other with the same horizontal scales, you can see how they correspond to each other.

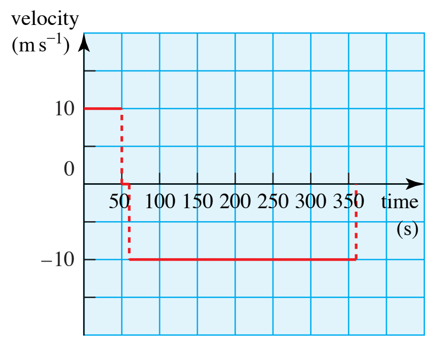


Figure 1.7

Distance–time graphs

Figure 1.8 is the distance–time graph of Amy's journey. It differs from the position–time graph because it shows how far she travels irrespective of her direction. There are no negative values.

The gradient of this graph represents Amy's speed rather than her velocity.

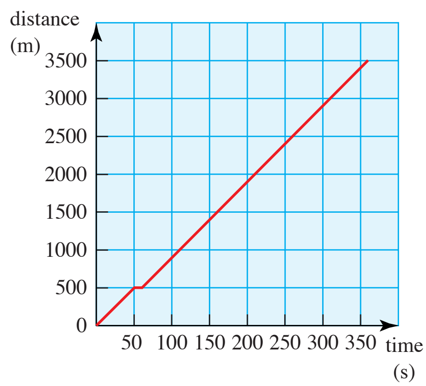


Figure 1.8

It has been assumed that Amy starts and stops instantaneously. What would more realistic graphs look like? Would it make a lot of difference to the answers if you tried to be more realistic?

Average speed and average velocity

You can find Amy's average speed on her way to college by using the definition

- $\text{average speed} = \dfrac{\text{total distance travelled}}{\text{total time taken}}$

When the distance is in metres and the time in seconds, speed is found by dividing metres by seconds and is written as m s^{-1}. So Amy's average speed is

$$\frac{3500\,\text{m}}{360\,\text{s}} = 9.72\,\text{m s}^{-1}$$

Amy's average velocity is different. Her displacement from start to finish is −2500 m so

The college is in the negative direction.

- $\text{average velocity} = \dfrac{\text{displacement}}{\text{time taken}}$

$$= \frac{-2500}{360}$$

$$= -6.94\,\text{m s}^{-1}$$

If Amy had taken the same time to go straight from home to college at a steady speed, this steady speed would have been $6.94\,\text{m s}^{-1}$.

Velocity at an instant

The position–time graph for a marble thrown straight up into the air at $5\,\mathrm{m\,s^{-1}}$ is curved because the velocity is continually changing.

The velocity is represented by the gradient of the position–time graph. When a position–time graph is curved like this you can find *the velocity at an instant* of time by drawing a tangent as in figure 1.9.

The velocity at P is approximately

$$\frac{0.6}{0.25} = 2.4\,\mathrm{m\,s^{-1}}$$

The velocity–time graph is shown in figure 1.10.

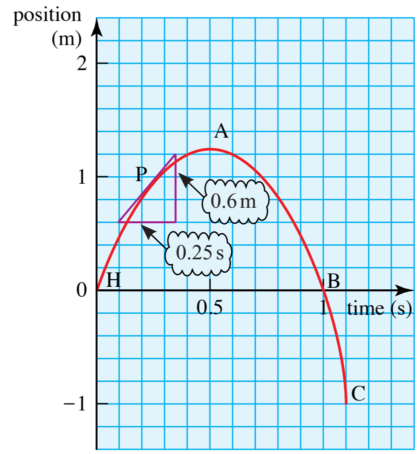


Figure 1.9

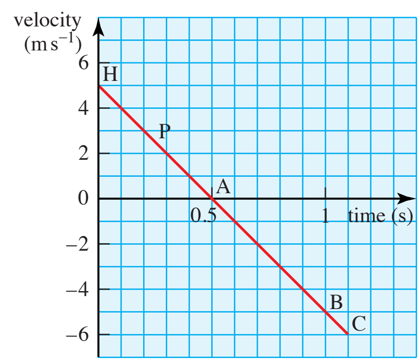


Figure 1.10

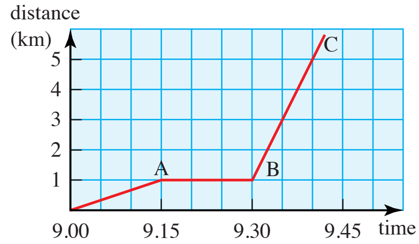
What is the velocity at H, A, B and C? The speed of the marble increases after it reaches the top. What happens to the velocity?

At the point A, the velocity and gradient of the position–time graph are zero. We say the marble is *instantaneously at rest*. The velocity at H is positive because the marble is moving in the positive direction (upwards). The velocity at B and at C is negative because the marble is moving in the negative direction (downwards).

EXERCISE 1B

1 Draw a speed–time graph for Amy's journey on page 6.

2 The distance–time graph shows the relationship between distance travelled and time for a person who leaves home at 9.00 am, walks to a bus stop and catches a bus into town.



(i) Describe what is happening during the time from A to B.

(ii) The section BC is much steeper than OA; what does this tell you about the motion?

(iii) Draw the speed–time graph for the person.

(iv) What simplifications have been made in drawing these graphs?

3 For each of the following journeys find

(a) the initial and final positions

(b) the total displacement

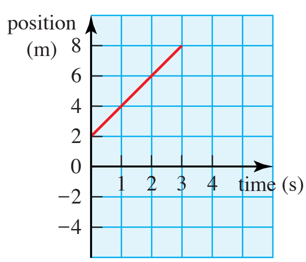
(c) the total distance travelled

(d) the velocity and speed for each part of the journey

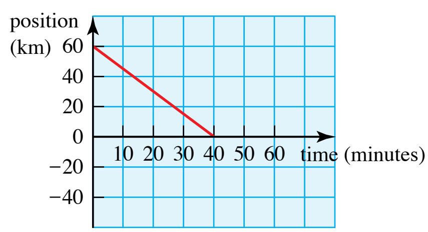
(e) the average velocity for the whole journey

(f) the average speed for the whole journey.

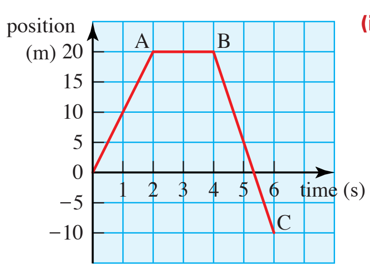
(i)



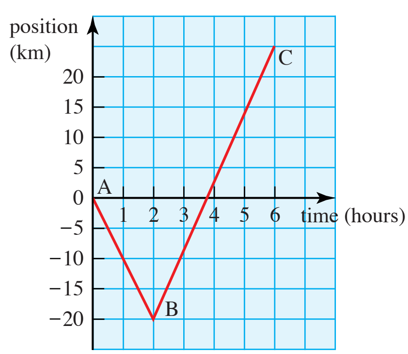
(ii)



(iii)



(iv)



4 A plane flies from London to Toronto, a distance of 5700 km, at an average speed of 1280 $\mathrm{km\,h^{-1}}$. It returns at an average speed of 1200 $\mathrm{km\,h^{-1}}$. Find the average speed for the round trip.

Acceleration

In everyday language, the word 'accelerate' is usually used when an object speeds up and 'decelerate' when it slows down. The idea of deceleration is sometimes used in a similar way by mathematicians but in mathematics the word *acceleration* is used whenever there is a change in velocity, whether an object is speeding up, slowing down or changing direction. Acceleration is *the rate at which the velocity changes*.

Over a period of time

- $$\text{average acceleration} = \frac{\text{change in velocity}}{\text{time taken}}$$

Acceleration is represented by the gradient of a velocity–time graph. It is a vector and can take different signs in a similar way to velocity. This is illustrated by Tom's cycle journey which is shown in figure 1.11.

Tom turns on to the main road at $4\,\mathrm{m\,s^{-1}}$, accelerates uniformly, maintains a constant speed and then slows down uniformly to stop when he reaches home.

Between A and B, Tom's velocity increases by $(10 - 4) = 6\,\mathrm{m\,s^{-1}}$ in 6 seconds, that is by 1 metre per second every second.

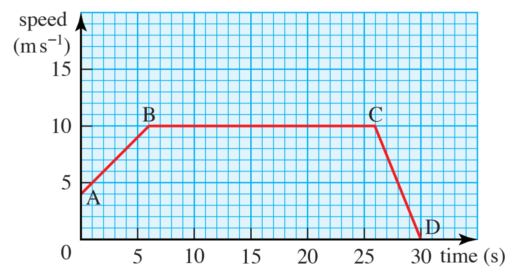


Figure 1.11

This acceleration is written as $1\,\mathrm{m\,s^{-2}}$ (one metre per second squared) and is the gradient of AB.

From B to C acceleration $= 0\,\mathrm{m\,s^{-2}}$ ← There is no change in velocity.

From C to D acceleration $= \dfrac{(0 - 10)}{(30 - 26)} = -2.5\,\mathrm{m\,s^{-2}}$

From C to D, Tom is slowing down while still moving in the positive direction towards home, so his acceleration, the gradient of the graph, is negative.

The sign of acceleration

Think again about the marble thrown up into the air with a speed of $5\,\mathrm{m\,s^{-1}}$.

Figure 1.12 represents the velocity when *upwards* is taken as the positive direction and shows that the velocity *decreases* from $+5\,\mathrm{m\,s^{-1}}$ to $5\,\mathrm{m\,s^{-1}}$ in 1 second.

This means that the gradient, and hence the acceleration, is *negative*. It is $-10\,\mathrm{m\,s^{-2}}$. (You might recognise the number 10 as an approximation to g. See Chapter 2 page 28.)

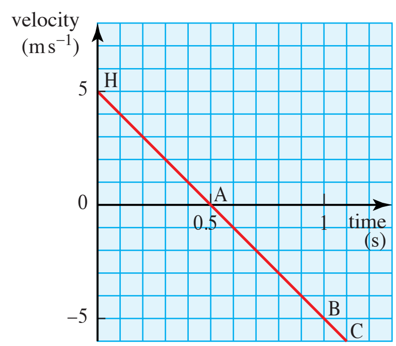


Figure 1.12

A car accelerates away from a set of traffic lights. It accelerates to a maximum speed and at that instant starts to slow down to stop at a second set of lights. Which of the graphs below could represent

(i) the distance–time graph
(ii) the velocity–time graph
(iii) the acceleration–time graph of its motion?

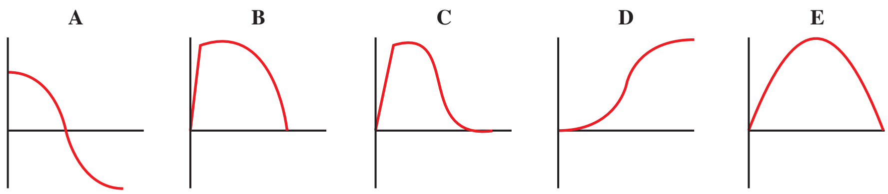
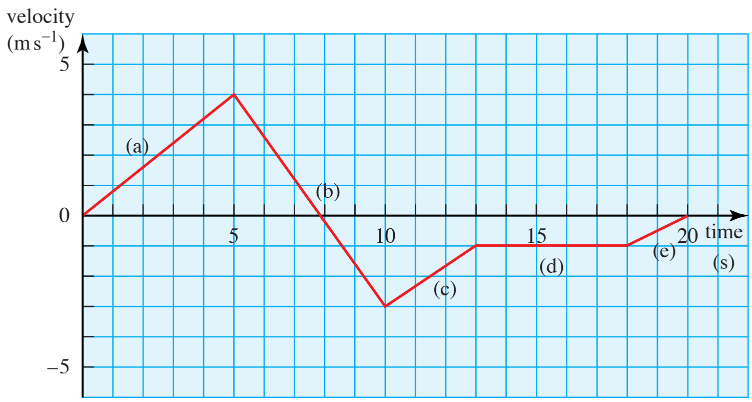


Figure 1.13

EXERCISE 1C

1 **(i)** Calculate the acceleration for each part of the following journey.



(ii) Use your results to sketch an acceleration–time graph.

2 A particle moves so that its position x metres at time t seconds is $x = 2t^3 - 18t$.

(i) Calculate the position of the particle at times $t = 0$, 1, 2, 3 and 4.
(ii) Draw a diagram showing the position of the particle at these times.
(iii) Sketch a graph of the position against time.
(iv) State the times when the particle is at the origin and describe the direction in which it is moving at those times.

3 A train takes 45 minutes to complete its 24 kilometre trip. It stops for 1 minute at each of 7 stations during the trip.

(i) Calculate the average speed of the train.
(ii) What would be the average speed if the stop at each station was reduced to 20 seconds?

4 When Louise is planning car journeys she reckons that she can cover distances along main roads at roughly 100 km h^{-1} and those in towns at 30 km h^{-1}.

(i) Find her average speed for each of the following journeys.

(a) 20 km on main roads and then 10 km in a town

(b) 150 km on main roads and then 2 km in a town

(c) 20 km on main roads and then 20 km in a town

(ii) In what circumstances would her average speed be 65 km h^{-1}?

5 A lift travels up and down between the ground floor (G) and the roof garden (R) of a hotel. It starts from rest, takes 5 s to increase its speed uniformly to 2 m s^{-1}, maintains this speed for 5 s and then slows down uniformly to rest in another 5 s. In the following questions, use upwards as positive.

(i) Sketch a velocity–time graph for the journey from G to R.

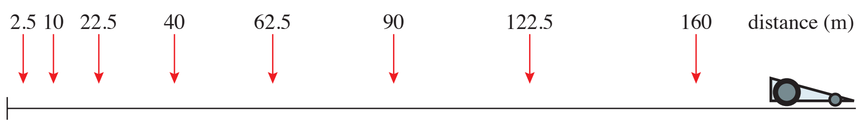
On one occasion the lift stops for 5 s at R before returning to G.

(ii) Sketch a velocity–time graph for this journey from G to R and back.

(iii) Calculate the acceleration for each 5 s interval. Take care with the signs.

(iv) Sketch an acceleration–time graph for this journey.

6 A film of a dragster doing a 400 m run from a standing start yields the following positions at 1 second intervals.



(i) Draw a displacement–time graph of its motion.

(ii) Use your graph to help you to sketch

(a) the velocity–time graph

(b) the acceleration–time graph.

Using areas to find distances and displacements

These graphs model the motion of a stone falling from rest.

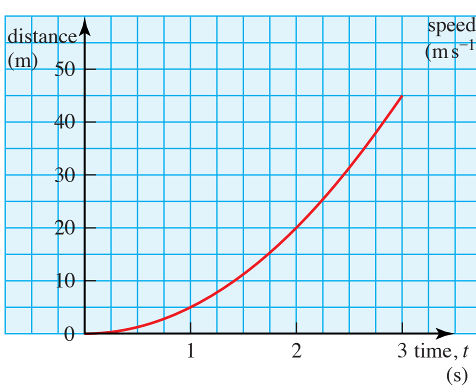


Figure 1.14

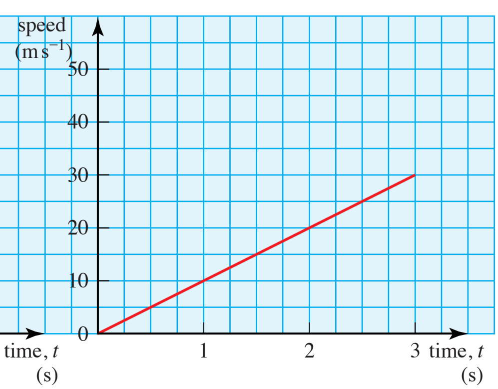


Figure 1.15

? Calculate the area between the speed–time graph and the time axis from

(i) $t = 0$ to 1 **(ii)** $t = 0$ to 2 **(iii)** $t = 0$ to 3.

Compare your answers with the distance that the stone has fallen, shown on the distance–time graph, at $t = 1$, 2 and 3. What conclusions do you reach?

- The area between a speed–time graph and the time axis represents the distance travelled.

There is further evidence for this if you consider the units on the graphs. Multiplying metres per second by seconds gives metres. A full justification relies on the calculus methods you will learn in Chapter 7.

Finding the area under speed–time graphs

Many of these graphs consist of straight-line sections. The area is easily found by splitting it up into triangles, rectangles or trapezia.

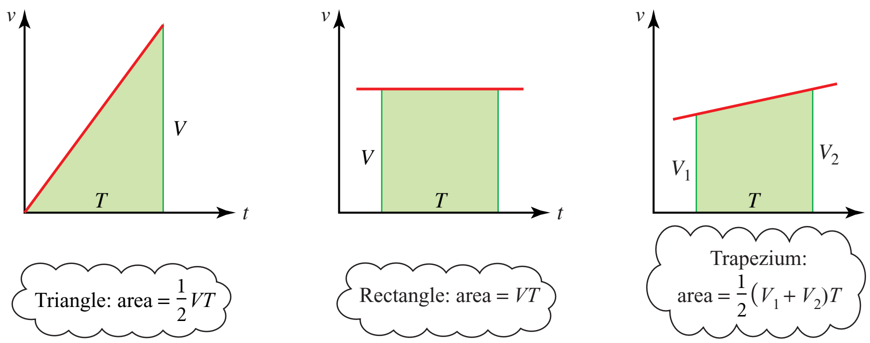


Figure 1.16

EXAMPLE 1.1

The graph shows Hinesh's journey from the time he turns on to the main road until he arrives home. How far does Hinesh cycle?

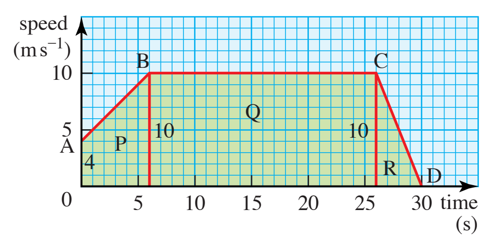


Figure 1.17

SOLUTION

Split the area under the speed–time graph into three regions.

P	trapezium:	area $= \frac{1}{2}(4 + 10) \times 6 =$	42 m
Q	rectangle:	area $= 10 \times 20$	= 200 m
R	triangle:	area $= \frac{1}{2} \times 10 \times 4$	= 20 m
		total area	= 262 m

Hinesh cycles 262 m.

? What is the meaning of the area between a velocity–time graph and the time axis?

The area between a velocity–time graph and the time axis

EXAMPLE 1.2

Sunil walks east for 6 s at 2 m s^{-1} then west for 2 s at 1 m s^{-1}. Draw

(i) a diagram of the journey
(ii) the speed–time graph
(iii) the velocity–time graph.

Interpret the area under each graph.

SOLUTION

(i) Sunil's journey is illustrated below.

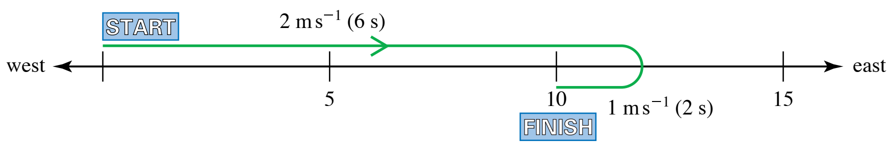


Figure 1.18

(ii) Speed–time graph

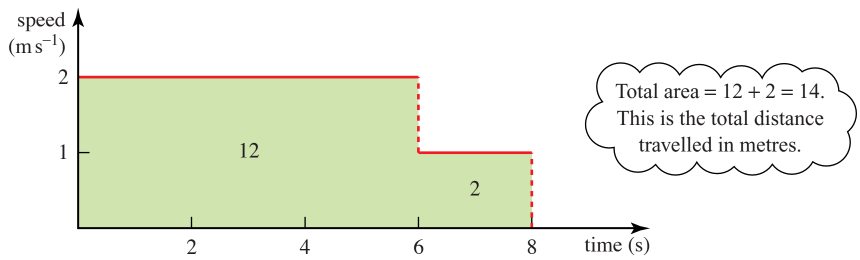


Figure 1.19

(iii) Velocity–time graph

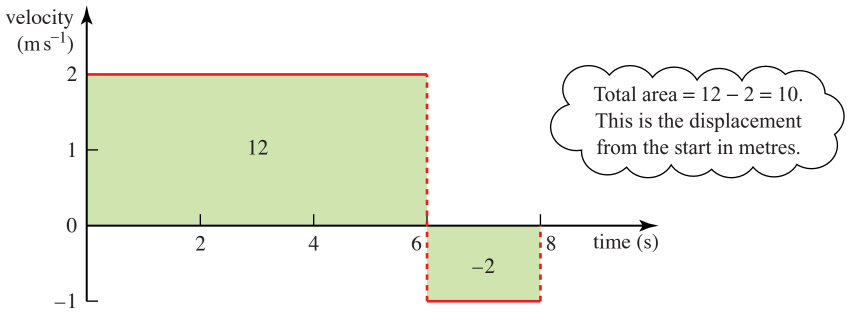


Figure 1.20

- The area between a velocity–time graph and the time axis represents the change in position, that is the displacement.

When the velocity is negative, the area is below the time axis and represents a displacement in the negative direction, west in this case.

Estimating areas

Sometimes the velocity–time graph does not consist of straight lines so you have to make the best estimate you can by counting the squares underneath it or by replacing the curve by a number of straight lines as for the trapezium rule (see *Pure Mathematics 2*, Chapter 5).

? This speed–time graph shows the motion of a dog over a 60 s period.

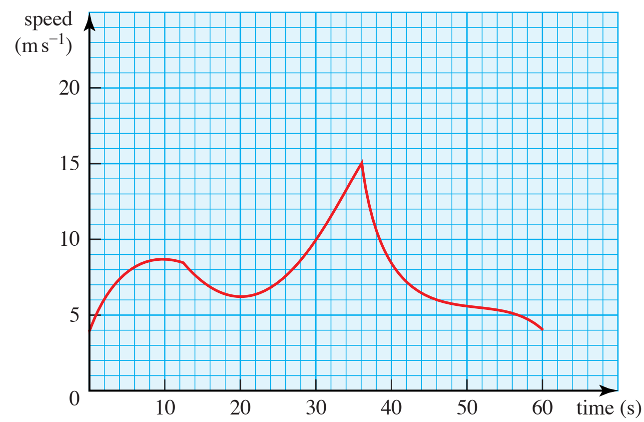


Figure 1.21

Estimate how far the dog travelled during this time.

EXAMPLE 1.3

On the London underground, Oxford Circus and Piccadilly Circus are 0.8 km apart. A train accelerates uniformly to a maximum speed when leaving Oxford Circus and maintains this speed for 90 s before decelerating uniformly to stop at Piccadilly Circus. The whole journey takes 2 minutes. Find the maximum speed.

SOLUTION

The sketch of the speed–time graph of the journey shows the given information, with suitable units. The maximum speed is $v\,\mathrm{m\,s^{-1}}$.

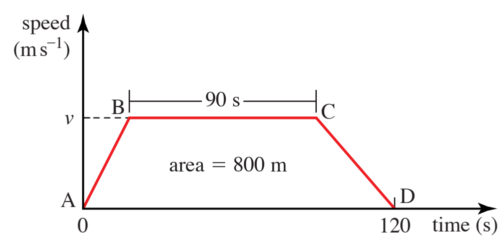


Figure 1.22

The area is $\frac{1}{2}(120 + 90) \times v = 800$

$$v = \frac{800}{105}$$

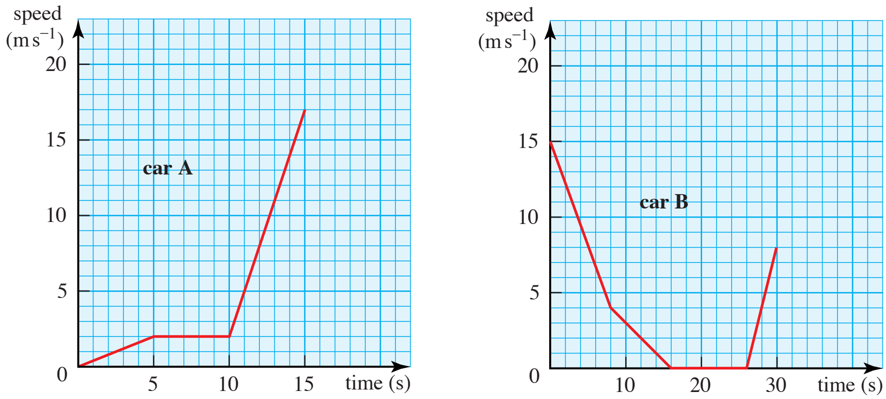
$$= 7.619$$

The maximum speed of the train is $7.6\,\mathrm{m\,s^{-1}}$ (to 2 s.f.).

? Does it matter how long the train takes to speed up and slow down?

EXERCISE 1D

1 The graphs show the speeds of two cars travelling along a street.



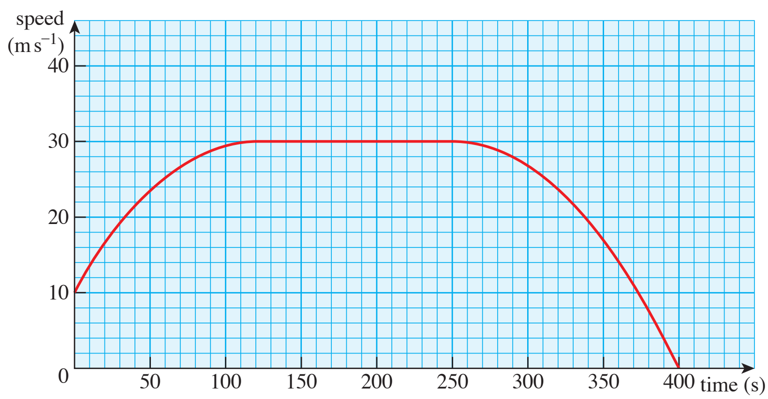
For each car find

(i) the acceleration for each part of its motion

(ii) the total distance it travels in the given time

(iii) its average speed.

2 The graph shows the speed of a lorry when it enters a very busy road.



(i) Describe the journey over this time.

(ii) Use a ruler to make a tangent to the graph and hence estimate the acceleration at the beginning and end of the period.

(iii) Estimate the distance travelled and the average speed.

3 A train leaves a station where it has been at rest and picks up speed at a constant rate for 60 s. It then remains at a constant speed of $17\,\mathrm{m\,s^{-1}}$ for 60 s before it begins to slow down uniformly as it approaches a set of signals. After 45 s it is travelling at $10\,\mathrm{m\,s^{-1}}$ and the signal changes. The train again increases speed uniformly for 75 s until it reaches a speed of $20\,\mathrm{m\,s^{-1}}$. A second set of signals then orders the train to stop, which it does after slowing down uniformly for 30 s.

(i) Draw a speed–time graph for the train.

(ii) Use your graph to find the distance that it has travelled from the station.

4 When a parachutist jumps from a helicopter hovering above an airfield her speed increases at a constant rate to $28\,\text{m}\,\text{s}^{-1}$ in the first 3 s of her fall. It then decreases uniformly to $8\,\text{m}\,\text{s}^{-1}$ in a further 6 s, remaining constant until she reaches the ground.

(i) Sketch a speed–time graph for the parachutist.

(ii) Find the height of the plane when the parachutist jumps out if the complete jump takes 1 minute.

5 A car is moving at $20\,\text{m}\,\text{s}^{-1}$ when it begins to increase speed. Every 10 s it gains $5\,\text{m}\,\text{s}^{-1}$ until it reaches its maximum speed of $50\,\text{m}\,\text{s}^{-1}$ which it retains.

(i) Draw the speed–time graph of the car.

(ii) When does the car reach its maximum speed of $50\,\text{m}\,\text{s}^{-1}$?

(iii) Find the distance travelled by the car after 150 s.

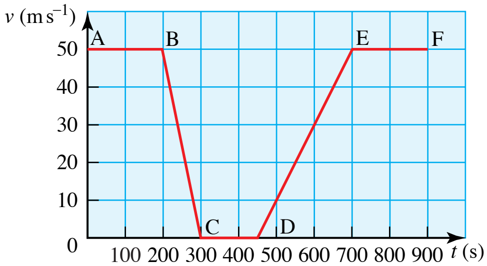
(iv) Write down expressions for the speed of the car t seconds after it begins to speed up.

6 A train takes 10 minutes to travel between two stations. The train accelerates at a rate of $0.5\,\text{m}\,\text{s}^{-2}$ for 30 s. It then travels at a constant speed and is finally brought to rest in 15 s with a constant deceleration.

(i) Sketch a velocity–time graph for the journey.

(ii) Find the steady speed, the rate of deceleration and the distance between the two stations.

7 A train was scheduled to travel at $50\,\text{m}\,\text{s}^{-1}$ for 15 minutes on part of its journey. The velocity–time graph illustrates the actual progress of the train which was forced to stop because of signals.



(i) Without carrying out any calculations, describe what was happening to the train in each of the stages BC, CD and DE.

(ii) Find the deceleration of the train while it was slowing down and the distance travelled during this stage.

(iii) Find the acceleration of the train when it starts off again and the distance travelled during this stage.

(iv) Calculate by how long the stop will have delayed the train.

(v) Sketch the distance–time graph for the journey between A and F, marking the points A, B, C, D, E and F. **[MEI]**

8 A car is travelling at $36\,\text{km}\,\text{h}^{-1}$ when the driver has to perform an emergency stop. During the time the driver takes to appreciate the situation and apply the brakes the car has travelled 7 m ('thinking distance'). It then pulls up with constant deceleration in a further 8 m ('braking distance') giving a total stopping distance of 15 m.

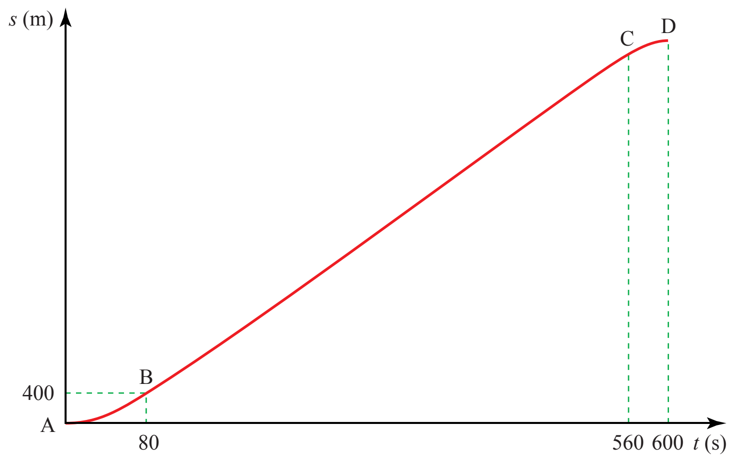
(i) Find the initial speed of the car in metres per second and the time that the driver takes to react.

(ii) Sketch the velocity–time graph for the car.

(iii) Calculate the deceleration once the car starts braking.

(iv) What is the stopping distance for a car travelling at $60\,\text{km}\,\text{h}^{-1}$ if the reaction time and the deceleration are the same as before?

9 The diagram shows the displacement–time graph for a car's journey. The graph consists of two curved parts AB and CD, and a straight line BC. The line BC is a tangent to the curve AB at B and a tangent to the curve CD at C. The gradient of the curves at $t = 0$ and $t = 600$ is zero, and the acceleration of the car is constant for $0 < t < 80$ and for $560 < t < 600$. The displacement of the car is 400 m when $t = 80$.



(i) Sketch the velocity–time graph for the journey.

(ii) Find the velocity at $t = 80$.

(iii) Find the total distance for the journey.

(iv) Find the acceleration of the car for $0 < t < 80$.

[**Cambridge AS & A Level Mathematics 9709, Paper 4 Q5 November 2005**]

10 A train travels from A to B, a distance of 20 000 m, taking 1000 s. The journey has three stages. In the first stage the train starts from rest at A and accelerates uniformly until its speed is $V\,\mathrm{m\,s^{-1}}$. In the second stage the train travels at constant speed $V\,\mathrm{m\,s^{-1}}$ for 600 s. During the third stage of the journey the train decelerates uniformly, coming to rest at B.

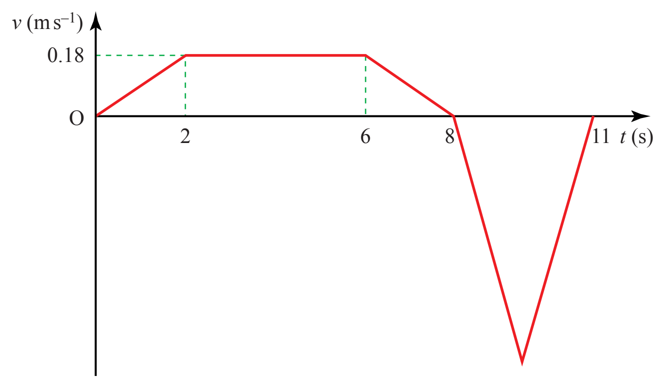
(i) Sketch the velocity–time graph for the train's journey.

(ii) Find the value of V.

(iii) Given that the acceleration of the train during the first stage of the journey is $0.15\,\mathrm{m\,s^{-2}}$, find the distance travelled by the train during the third stage of the journey.

[**Cambridge AS & A Level Mathematics 9709, Paper 4 Q6 November 2008**]

11 The diagram shows the velocity–time graph for the motion of a machine's cutting tool. The graph consists of five straight line segments. The tool moves forward for 8 s while cutting and then takes 3 s to return to its starting position.



Find

(i) the acceleration of the tool during the first 2 s of the motion,

(ii) the distance the tool moves forward while cutting,

(iii) the greatest speed of the tool during the return to its starting position.

[**Cambridge AS & A Level Mathematics 9709, Paper 41 Q2 June 2010**]

INVESTIGATION

Train journey

If you look out of a train window in many countries you will see distance markers beside the track (in the UK they are every quarter of a mile). Take a train journey and record the time as you go past each marker. Use your figures to draw distance–time, speed–time and acceleration–time graphs. What can you conclude about the greatest acceleration, deceleration and speed of the train?

KEY POINTS

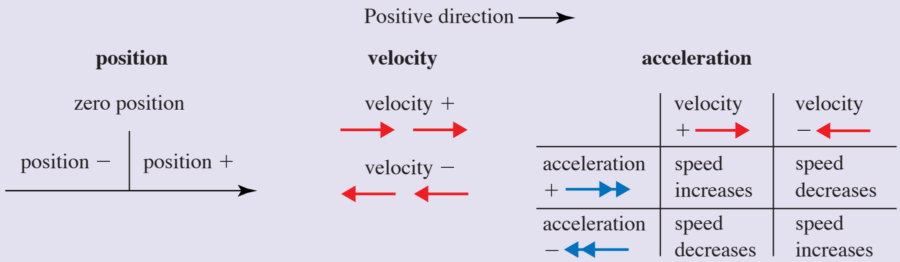
1 **Vectors** (with magnitude and direction) / **Scalars** (magnitude only)

Vectors (with magnitude and direction)	Scalars (magnitude only)
Displacement	Distance
Position – displacement from a fixed origin	
Velocity – rate of change of position	Speed – magnitude of velocity
Acceleration – rate of change of velocity	
	Time

- *Vertical* is towards the centre of the earth; *horizontal* is perpendicular to vertical.

2 **Diagrams**

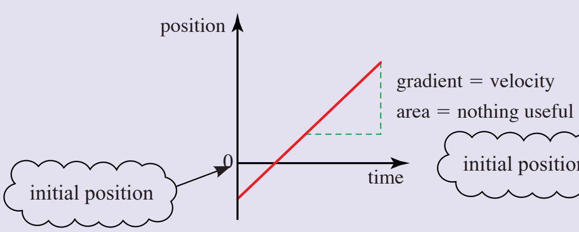
- Motion along a line can be illustrated vertically or horizontally (as shown).



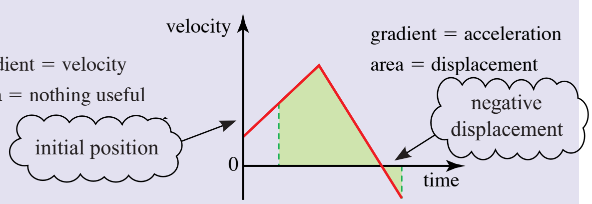
- $\text{Average speed} = \dfrac{\text{total distance travelled}}{\text{total time taken}}$
- $\text{Average velocity} = \dfrac{\text{displacement}}{\text{time taken}}$
- $\text{Average acceleration} = \dfrac{\text{change in velocity}}{\text{time taken}}$

3 **Graphs**

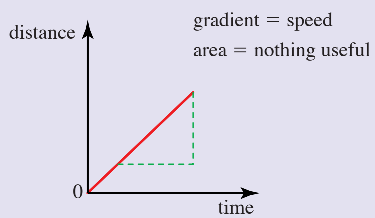
- *Position–time*



- *Velocity–time*



- *Distance–time*



- *Speed–time*